XXIV. *On the Tangential of a Cubic.* By ARTHUR CAYLEY, *Esq., F.R.S.*

Received February 11,—Read March 18, 1858.

IN my "Memoir on Curves of the Third Order*," I had occasion to consider a derivative which may be termed the "tangential" of a cubic, viz. the tangent at the point (x, y, z) of the cubic curve $(*\between x, y, z)^3=0$ meets the curve in a point (ξ, η, ζ), which is the tangential of the first-mentioned point; and I showed that when the cubic is represented in the canonical form $x^3+y^3+z^3+6lxyz=0$, the coordinates of the tangential may be taken to be $x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3)$. The method given for obtaining the tangential may be applied to the general form $(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3$: it seems desirable, in reference to the theory of cubic forms, to give the expression of the tangential for the general form†; and this is what I propose to do, merely indicating the steps of the calculation, which was performed for me by Mr. CREEDY.

The cubic form is

$$(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3,$$

which means

$$ax^3+by^3+cz^3+3fy^2z+3gz^2x+3hx^2y+3iyz^2+3jzx^2+3kxy^2+6lxyz;$$

and the expression for ξ is obtained from the equation

$$x^2\xi=(b, f, i, c\between (j, f, c, i, g, l\between x, y, z)^2, -(h, b, i, f, l, k\between x, y, z)^2)^3$$
$$-(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3(\mathfrak{C}x+\mathfrak{D}),$$

where the second line is in fact equal to zero, on account of the first factor, which vanishes. And $\mathfrak{C}$, $\mathfrak{D}$ denote respectively quadric and cubic functions of (y, z), which are to be determined so as to make the right-hand side divisible by x^2; the resulting value of ξ may be modified by the adjunction of the evanescent term

$$(2x+hy+gz)(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3,$$

where a, g, h are arbitrary coefficients; but as it is not obvious how these coefficients should be determined in order to present the result in the most simple form, I have given the result in the form in which it was obtained without the adjunction of any such term.

Write for shortness

$$\mathrm{P}=(k, l \between y, z),$$
$$\mathrm{Q}=(b, f, i \between y, z)^2,$$

* Philosophical Transactions, vol. cxlvii. 1857.

† At the time when the present paper was written, I was not aware of Mr. SALMON's theorem (Higher Plane Curves, p. 156), that the tangential of a point of the cubic is the intersection of the tangent of the cubic with the first or line polar of the point with respect to the Hessian; a theorem, which at the same time that it affords the easiest mode of calculation, renders the actual calculation of the coordinates of the tangential less important. Added 7th October, 1858.—A. C.

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$$
\begin{aligned}
\mathrm{R}&=(l,\ g, \qquad \between y,\ z)\,,\\
\mathrm{S}&=(f,\ i,\ c \ \between y,\ z)^2,\\
\mathrm{B}&=(h,\ j \qquad \between y,\ z)\,,\\
\mathrm{C}&=(k,\ l,\ g \ \between y,\ z)^2,\\
\mathrm{D}&=(b,\ f,\ i,\ c\between y,\ z)^3,
\end{aligned}
$$

so that

$$
\begin{aligned}
&(h,\ b,\ i,\ f,\ l,\ k &&\between x,\ y,\ z)^2=(h\ ,\ \mathrm{P},\ \mathrm{Q} &&\between x,\ 1)^2,\\
&(j,\ f,\ c,\ i,\ g,\ l &&\between x,\ y,\ z)^2=(j\ ,\ \mathrm{R},\ \mathrm{S} &&\between x,\ 1)^2,\\
&(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l &&\between x,\ y,\ z)^3=(a\ ,\ \mathrm{B},\ \mathrm{C},\ \mathrm{D} &&\between x,\ 1)^3.\\
&\mathfrak{C}x+\mathfrak{D} && \qquad\qquad\quad =(\mathfrak{C},\ \mathfrak{D} &&\between x,\ 1),
\end{aligned}
$$

and then for greater convenience writing $(h,\ 2\mathrm{P},\ \mathrm{Q}\between x,\ 1)^2$, &c. for $(h,\ \mathrm{P},\ \mathrm{Q}\between x,\ 1)^2$, &c., and omitting the $(x,\ 1)^2$, &c. and the arrow-heads, or representing the functions simply by $(h,\ 2\mathrm{P},\ \mathrm{Q})$, &c., we have

$$
\begin{aligned}
x^2\xi=\ & b(j,\ 2\mathrm{R},\ \mathrm{S})^3\\
&-3f(j,\ 2\mathrm{R},\ \mathrm{S})^2.(h\ ,\ 2\mathrm{P},\ \mathrm{Q})\\
&+3i\,(j,\ 2\mathrm{R},\ \mathrm{S})\,.(h\ ,\ 2\mathrm{P},\ \mathrm{Q})^2\\
&-\ c \qquad\qquad\qquad\quad .(h\ ,\ 2\mathrm{P},\ \mathrm{Q})^3\\
&-\ (a,\ 3\mathrm{B},\ 3\mathrm{C},\ \mathrm{D})\,.(\mathfrak{C},\ \mathfrak{D}\ ),
\end{aligned}
$$

which can be developed in terms of the quantities which enter into it. The conditions, in order that the coefficients of x, x^0 may vanish, are thus seen to be

$$\mathrm{D}\mathfrak{D}=b\mathrm{S}^3-3f\mathrm{S}^2\mathrm{Q}+3i\mathrm{S}\mathrm{Q}^2-c\mathrm{Q}^3,$$

$$\mathrm{D}\mathfrak{C}-3\mathrm{C}\mathfrak{D}=b(6\mathrm{R}\mathrm{S}^2)-3f(2\mathrm{S}^2\mathrm{P}+4\mathrm{R}\mathrm{S}\mathrm{Q})+3i(2\mathrm{R}\mathrm{Q}^2+4\mathrm{S}\mathrm{P}\mathrm{Q})-c6\mathrm{P}\mathrm{Q}^2,$$

and from these we obtain

$\mathfrak{C}=($

$-3\ bck$ $+6\ bil$ $+3\ fik$ $-6\ f^2l$	$+6\ big$ $-6\ cfk$ $-6\ f^2g$ $+6\ i^2k$	$+3\ bcg$ $-6\ cfl$ $-3\ fgi$ $+6\ i^2l$

$\between y,\ z)^2$

$\mathfrak{D}=($

$-1\ b^2c$ $+3\ bfi$ $-2\ f^3$	$-3\ bcf$ $+6\ bi^2$ $-3\ f^2i$	$+3\ bci$ $-6\ cf^2$ $+3\ fi^2$	$+1\ bc^2$ $-3\ cfi$ $+2\ i^3$

$\between y,\ z)^3$

and substituting these values, the right-hand side of the equation divides by x^2, and throwing out this factor we have the value of ξ; and the values of η, ζ may be thence deduced by a mere interchange of letters. The value for ξ is

x^4	x^3y	x^3z	x^2y^2	x^2yz	x^2z^2	xy^3	xy^2z	xyz^2	xz^3	y^4	y^3z	y^2z^2	yz^3	z^4
$+1\,bj^3$	$+6\,bj^2l$	$+6\,bgj^2$	$+3\,abck$	$-6\,abgi$	$-3\,abcg$	$+1\,ab^2c$	$+3\,abcf$	$-3\,abci$	$-1\,abc^2$	$+3\,b^2ij$	$+3\,b^2cj$	$+9\,bcfj$	$-3\,bc^2h$	$+3\,bcg^2$
$-1\,ch^3$	$-6\,ch^2k$	$-6\,ch^2l$	$-6\,abil$	$+6\,acfk$	$+6\,acfl$	$-3\,abfi$	$-6\,abi^2$	$+6\,acf^2$	$+3\,acfi$	$-3\,bck^2$	$-3\,bcfh$	$-9\,bchi$	$+6\,bcgl$	$-3\,c^2fh$
$-3\,fhj^2$	$-12\,fhjl$	$-12\,fghj$	$+6\,af^2l$	$+6\,af^2g$	$+3\,afgi$	$+2\,af^3$	$+3\,af^2i$	$-3\,afi^2$	$-2\,ai^3$	$-3\,bf^2j$	$-3\,bckl$	$+18\,bgil$	$+3\,bcij$	$-6\,cfgl$
$+3\,h^2ij$	$-6\,fj^2k$	$-6\,fj^2l$	$-3\,afik$	$-6\,ai^2k$	$-6\,ai^2l$	$-3\,bchk$	$-12\,bchl$	$-9\,bcgh$	$+3\,bcgj$	$-3\,bfhi$	$+3\,bfij$	$-18\,cfkl$	$+6\,bg^2i$	$+3\,cfij$
	$+6\,h^2il$	$+6\,gh^2i$	$-3\,bch^2$	$+24\,bgjl$	$+3\,bcj^2$	$-6\,bhil$	$+9\,bcjk$	$+12\,bcjl$	$+8\,bg^3$	$+6\,bikl$	$+3\,bgik$	$-18\,f^2gl$	$+6\,cf^2j$	$+3\,chi^2$
	$+12\,hijk$	$+12\,hijl$	$+6\,bhij$	$+6\,bij^2$	$+12\,bg^2j$	$+12\,bijk$	$-6\,bghi$	$+24\,bg^2l$	$-12\,cfgh$	$+3\,f^3h$	$-6\,bhi^2$	$-9\,f^2ij$	$-6\,cfgk$	$-3\,fg^2i$
			$+12\,bjl^2$	$-6\,cfh^2$	$-6\,cfhj$	$+8\,bl^3$	$+24\,bgl^2$	$+6\,bgij$	$+3\,cfjl$	$-6\,f^2kl$	$+12\,bil^2$	$+9\,fhi^2$	$-3\,cfhi$	$+6\,gi^2l$
			$-12\,chk^2$	$-24\,chkl$	$-24\,fjl^2$	$-8\,ck^3$	$+18\,bijl$	$-18\,cfhl$	$-8\,cl^3$	$+3\,fik^2$	$-6\,cfk^2$	$+18\,i^2kl$	$-12\,cfl^2$	$-3\,i^3j$
			$-6\,f^2hj$	$-6\,f^2j^2$	$-12\,fg^2h$	$+6\,f^2hl$	$-6\,cfhk$	$+6\,cfjk$	$-24\,fg^2l$		$-6\,f^3j$		$-6\,f^2g^2$	
			$-3\,fh^2i$	$-24\,fghl$	$-24\,fgjl$	$-12\,f^2jk$	$-24\,ck^2l$	$-24\,ckl^2$	$-3\,fgij$		$-6\,f^2gk$		$-6\,fgil$	
			$-12\,fhl^2$	$-24\,fgjk$	$-3\,fij^2$	$+3\,fhik$	$+6\,f^2gh$	$-6\,f^2gj$	$+12\,ghi^2$		$+9\,f^2hi$		$-9\,fi^2j$	
			$-24\,fjkl$	$-24\,fjl^2$	$+24\,ghil$	$-24\,fkl^2$	$-18\,f^2jl$	$-24\,fg^2k$	$+24\,gil^2$		$-12\,f^2l^2$		$+6\,gi^2k$	
			$+24\,hikl$	$+24\,ghik$	$+6\,hi^2j$	$+24\,ik^2l$	$-48\,fghl$	$+9\,fghi$	$-6\,i^2jl$		$+6\,fikl$		$+6\,hi^3$	
			$+12\,ijk^2$	$+6\,h^2i^2$	$+12\,ijl^2$		$+12\,fhil$	$-48\,fgl^2$			$+6\,i^2k^2$		$+12\,i^2l^2$	
				$+24\,hil^2$			$-9\,fijk$	$-12\,fijl$						
				$+24\,ijkl$			$-24\,fl^3$	$+48\,gikl$						
							$+24\,gik^2$	$+18\,hi^2l$						
							$+6\,h^2k$	$-6\,i^2jk$						
							$+48\,ikl^2$	$+24\,il^3$						

And it is not necessary to write down the corresponding values for η, ζ.